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THE MATHEMATICS STUDENT

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ON CIRCULAR CYLINDERS CIRCUMSCRIBING A TETRAHEDRON.

BY V. RAMASWAMI AIYAR.

1. The problem of determining all the circular cylinders circumscribing a tetrahedron ABCD occurred to the writer in a certain connection, but proved to be a rather tough problem. The writer was not able, by methods of pure geometry, to reduce it to known theorems. But the theorem underlying the problem was soon spotted and turned into other geometrical forms in which it could be more easily tackled analytically. This analysis was tough work and some little smartness was required for carrying it out successfully at a point where matters looked hopeless. It is proposed to give here a general account of this research ; and the writer hopes that some interested reader may be able to find out a comparatively simpler proof of the underlying theorem.

2. With reference to a tetrahedron ABCD, by "isogonal conjugates" we shall mean a pair of points P, P′ such that the normal co-ordinates of one are reciprocals of the normal co-ordinates of the other. The principal theorem can now be stated thus :—

If O be a point at infinity whose isogonal conjugate O′ also lies at infinity, then, the lines AO, BO, CO, DO are generators of a circular cylinder ... (2.1)

It follows that AO′, BO′, CO′, DO′ are also generators of a circular cylinder. The circumscribing circular cylinders thus occur in pairs.

3. Let the areas of the faces of the tetrahedron be denoted by A, B, C, D. If a point O, of normal co-odinates $\alpha, \beta, \gamma, \delta$ lies at infinity, we have

$$A\alpha + B\beta + C\gamma + D\delta = 0 \qquad \ldots \quad (3.1)$$

Its isogonal conjugate O' will be the point $1/\alpha$, $1/\beta$, $1/\gamma$, $1/\delta$. If this also lies at infinity, we should have

$$\frac{A}{\alpha} + \frac{B}{\beta} + \frac{C}{\gamma} + \frac{D}{\delta} = 0;$$

that is,

$$A\beta\gamma\delta + B\gamma\delta\alpha + C\delta\alpha\beta + D\alpha\beta\gamma = 0 \qquad \ldots \quad (3.2)$$

The point O is thus any point of the cubic curve, Γ, in which the cubic surface (3.2) cuts the plane at infinity (3.1).

Taking now a fixed point P, if we draw lines through P to all the points of the cubic curve Γ, we shall have a cubic cone, Γ_P, of vertex P; and we infer that the axes of circular cylinders circumscribing ABCD are parallel to generators of the cubic cone Γ_P.

4. Another way of stating the underlying theorem is as follows: —

Taking a sphere, centre at any point P, let α, β, γ, δ be great circles of the sphere whose planes are parallel to the faces of the tetrahedron ABCD.

If now O, O′ are foci of any sphero-conic inscribed in the spherical quadrilateral formed by α, β, γ, δ, *then, lines drawn through the vertices of the tetrahedron ABCD parallel to PO (or parallel to PO′) are generators of a circular cylinder* ... (4.1)

Yet another way of stating the theorem is as follows :—

Through any point P draw planes α, β, γ, δ *parallel to the faces of the tetrahedron ABCD. If now any point O is such that the feet of the perpendiculars from it on* α, β, γ, δ *lie in a plane X, then, lines drawn through A, B, C, D parallel to PO are generators of a circular cylinder* ... (4.2)

We also have the following :—

The perpendiculars drawn through A, B, C, D to the plane X are generators of a circular cylinder ... (4.3)

The form (4.2) is perhaps the simplest form of stating the under lying theorem.

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5. There is a theorem worth noting that forms the connecting link between the different forms of the underlying theorem given in the last para.

Let P be the vertex and PL, PL′ the focal lines of a quadric cone V. *Then, if S be any point in PL, the locus of the feet of the perpendiculars from S on tangent planes to the cone V is a circle X, whose plane is perpendicular to the focal line PL′* ... (5.1)

This may be called the pedal circle of S.

We add the following for enabling a fuller view of this theorem. If S′ be a point in PL′, the locus of the feet of the perpendiculars drawn from S′ on tangent planes to the cone V is a circle X′, whose plane is perpendicular to PL. This is the pedal circle of S′. Then we have the theorem that

The pedal circles X and X′ lie on a sphere whose centre is the middle point of SS′ ... (5.2)

6. We proceed now to some detailed consideration of the cubic curve Γ at infinity and the corresponding cubic cone Γ_P of vertex P.

The cubic Γ, as we saw, consists of points O, O′, which are isogonal conjugates and will generally be distinct. The cubic will usually be a bipartite or unipartite curve, and *not have* a double point. But if the faces of the tetrahedron are connected by one of the relations

$$D + A = B + C \qquad \ldots \quad (6.1)$$

$$D + B = C + A \qquad \ldots \quad (6.2)$$

$$D + C = A + B \qquad \ldots \quad (6.3)$$

then a double point occurs on the cubic by the coincidence of one pair of isogonal conjugates on it. Thus, in the case (6.1), the double point is $(1, -1, -1, 1)$, in the case (6.2), it is $(-1, 1, -1, 1)$, and in (6.3), it is $(-1, -1, 1, 1)$. *The condition for the occurrence of a double point on the cubic* Γ is really

$$U \equiv (D + A - B - C)(D + B - C - A)(D + C - A - B) = 0 \qquad \ldots \quad (6.4)$$

Though in U, as above written, there is a special treatment of D, yet this is only apparent, and the function is symmetric in A, B, C, D.

Pursuing the point further, it may be deduced that *the condition for the curve Γ to be unipartite is $U > 0$; and the condition for its being bipartite is $U < 0$* ... (6.5)

Correspondingly, the cone Γ_P will be *unipartite* if $U > 0$; *bipartite*, if $U < 0$, and *nodal* if $U = 0$ ... (6.6)

7. If the tetrahedron be such that two of the conditions (6.1), (6.2), (6.3) hold simultaneously, then the cubic Γ will have *two* double points; which means, it breaks up into the line at infinity joining these points and a conic at infinity passing through them. The cubic cone Γ_P correspondingly breaks into a plane and a cone of degree 2.

Thus if the two conditions holding be (6.2) and (6.3), that is,

$$\left.\begin{array}{l} D + B = C + A \\ D + C = A + B \end{array}\right\}$$

these give $B = C$ and $A = D$. Using these in para 3, we easily get at the following result:—

If, in the tetrahedron ABCD, the faces meeting along the edge AD are equal in area and likewise those meeting along the edge BC, then, the generators of circular cylinders circumscribing the tetrahedron fall into two sets; in one set the generators are parallel to the medial plane of the tetrahedron which bisects the edges AB, AC, DB, BC; in the other set, the generators are parallel to generators of the asymptotic cone of the hyperboloid $\alpha\delta = \beta\gamma$ ... (7.1)

8. If the tetrahedron be such that all the three conditions (6.1), (6,2), (6.3) hold, we should have $A = B = C = D$. When the areas of the faces are equal, the tetrahedron should be isosceles, that is, one whose opposite edges are equal. In this case, then, Γ breaks up into three lines at infinity and Γ_P breaks up into three planes; and we have the following result:

The generators of circular cylinders circumscribing an isosceles tetrahedron fall into three sets, the generators in each set being parallel to one of the medial planes of the tetrahedron ... (8.1)

The reader may find that the three medial planes of an isosceles tetrahedron are perpendicular to one another.

9. Let A, B, C, D be four co-planar points forming the vertices of a convex quadrangle ABCD. Let us seek to know all the circular

cylinders circumscribing the quadrangle. If we treat the quadrangle as an extreme case of a tetrahedron, the generators are parallel, as we saw, to those of a cubic cone. But, drawing parallels in any direction in the plane of the quadrangle itself, these can be regarded as generators of an immensely big circular cylinder constituted by the plane itself. Hence the cubic cone of the general case should break up into a plane (parallel to the plane of the quadrangle) and a cone of the second degree. Deleting the former, we see that

The generators of circular cylinders passing through four coplanar points forming a convex quadrangle are parallel to the generators of a cone of the second degree ... (9.1)

A special case arises when A, B, C, D are four concyclic points. In this case the cone of the second degree breaks up into two planes. Let the diagonals AC, BD intersect in O; and let OX, OY bisect the angles at O made by them. Draw OZ perpendicular to the plane of ABCD. We have the result:

The generators of circular cylinders circumscribing the four concylic points A, B, C, D fall into two sets; one set being lines parallel to the plane XOZ, and the other set being parallel to the plane YOZ ... (9.2)

10. A notable point both in connection with a tetrahedron ABCD and a convex quadrangle ABCD is as follows:

For every tetrahedron ABCD we can find an infinite number of others $A'B'C'D'$, of different forms, such that for any circular cylinder circumscribing the one there are circular cylinders, **with axes in the same direction**, *circumscribing the others* ... (10.1)

A like property holds for quadrangles also. In particular,

For every convex quadrangle ABCD in a plane we can assign a rhombus *$A'B'C'D'$ in the plane, such that for any circular cylinder circumscribing the one, there is a circular cylinder, with axis in the same direction, circumscribing the other* ... (10.2)

In this case, if A, B, C, D be concyclic, the rhombus $A'B'C'D'$ becomes a square.

CONTINUOUS GROUPS AND TWO THEOREMS OF EULER.

By D. D. Kosambi.

In the present note, I show how the theory of continuous groups is applicable to the discussion of Euler's theorem (1755) on homogeneous functions and to the equation of continuity in hydro-dynamics, also due to Euler.

All operations take place in a continuous manifold of the n variables $x^1, x^2, \ldots x^n$. The notation $f_{,i} = \partial f / \partial x^i$ is used for partial differentiation, along with the tensor summation convention for a repeated or "dummy" index : $A^r B_r = A^1 B_1 + A^2 B_2 + \ldots + A^n B_n$. The first section gives known results which are used as lemmas in the other two sections.

1. A continuous one parameter group of transformations is said to be generated by a vector field $u^i(x)$ or by the operator $D = u^r \partial / \partial x^r$ or by the infinitesimal transformation $\bar{x}^i = x^i + u^i \delta\tau$. The finite transformations in terms of a parameter t are given by

$$\bar{x}^i = x^i + tu^i + \frac{t^2}{2!} Du^i + \ldots \frac{t^n}{n!} D^{n-1} u^i + \ldots = \sum_{k=0}^{\infty} \frac{t^k}{k!} D^k (x^i) \quad \ldots \quad (1)$$

We shall have to distinguish between such transformations and point-transformations or changes of co-ordinates denoted by $x \longleftrightarrow x'$. A function of the co-ordinates that retains its value at corresponding points under non-singular point transformations is called an absolute invariant, or tensor of rank zero. A contravariant or velocity-like vector λ^i and a covariant or force-like vector μ_i transform thus :

$$\lambda' = \lambda^r \frac{\partial x'^i}{\partial x^r} \qquad \mu'_i = \mu_r \frac{\partial x^r}{\partial x'^i} \qquad \left| \frac{\partial x'^i}{\partial x^j} \right| \neq 0 \quad \ldots \quad (2)$$

Sets of functions that behave like the algebraic product of any number of vectors of each kind are called tensors. Some functions,

however, cannot be brought under this scheme, having the transformation :

$$F(x') = F(x) \left\{ \left| \frac{\partial x'^i}{\partial x^j} \right| \right\}^p \qquad \dots \quad (3)$$

under change of co-ordinates. These are called relative invariants of weight p.

The generators u^i of the one parameter group (1) form a contravariant vector field. Absolute invariants have the law of transformation under (1) given by

$$\bar{f} = f + t\,Df + \frac{t^2}{2!} D^2 f + \dots = \sum_{k=0}^{\infty} \frac{t^k}{k!} D^k f \qquad \dots \quad (4)$$

If $\bar{f} = f$, *i.e.*, $Df = 0$, f is said to be an *invariant of the group.* We shall consider relative invariants in the third section.

2. If under the group (1), a function is transformed as

$$\bar{f} = \psi(f)\ \phi(t) \qquad \dots \quad (5)$$

we shall say that the function has a multiplicative transform. With this definition, it is obvious that

The only multiplicative transforms are of type $\bar{f} = fe^{ct}$. *A necessary and sufficient condition thereto is that* f *be a solution of*

$$Df = cf \qquad \dots \quad (6)$$

Moreover,

The only groups under which the co-ordinates themselves have multiplicative transforms are those given by

$$\bar{x}^i = x^i e^{c_i t}.$$

The two results can be combined. After putting $\lambda = e^t$, we have $f(\lambda c_1 x^1 \dots \lambda c_m x^n) = \psi(f)\ \phi(\lambda)$ *is possible only when* $\psi\phi = \lambda^c f$. *A necessary and sufficient condition thereto is*

$$c_1 x^1 \frac{\partial f}{\partial x^1} + c_2 x^2 \frac{\partial f}{\partial x^2} + \dots + c_n x^n \frac{\partial f}{\partial x^n} = cf.$$

For $c_1 = c_2 = c_3 = \dots = c_n$, this is Euler's theorem on homogeneous functions. Let us attempt another generalization by searching

for functions with transforms under (1) of the type $\bar{f} = \psi(f, t)$ Thereby, $\psi(f, 0) = f$. Also, $\psi_t(f, 0) = Df$ must be a function of f alone: $Df = \phi(f)$. Consider the function H, defined by $H'\phi = H$. Then $H(f)$ has necessarily the transform

$$\bar{H} = He^t \qquad \phi(f) = Df = \frac{H}{H'}, \neq 0 \qquad \ldots \quad (7)$$

Therefore, *If a transform of the type* $\bar{f} = \psi(f, t)$ *is possible, then either* f *is an invariant of the group,* $(\psi_t = 0)$, *or there exist functions of* f *with a multiplicative transform, one such being given* by $H(f)$ where $\log H(y) = \int dy / \phi(y)$. $\phi(f) = Df \neq 0$.

For Euler's theorem, the corresponding result naturally reads:

If $f(\lambda x) = \phi(f, \lambda)$ *is possible, either* f *is homogeneous of degree null in* x^i *or there exist functions of* f *homogeneous of any degree. One of the first degree, moreover, is given by* $H(f)$ *where* $H/H' = Df \neq 0$.

This is about as much as can be done to generalise either the concept of homogeneous functions, or Euler's theorem. Here, Df has the special form $x^r f_{,r}$.

3. We come now to group transformation of some functions that are relative invariants under point transformation. The fundamental theorem is:

The Jacobian of the finite transformation (1) *is given by*

$$J = \left| \frac{\partial \bar{x}^i}{\partial x^j} \right| = 1 + \theta t + \frac{t^2}{2!}(D\theta + \theta^2) + \ldots = \sum_{k=0}^{\infty} \frac{t^k}{k!} \Theta^k (1) \ldots \quad (8)$$

where $\theta = u^r_{,r}$ *and** $\Theta = D + \theta$.

Herewith, the necessary convergence and differentiability conditions must be assumed. The theorem may be proved by direct computation, though I give another proof by induction.

---

* Θ^k does not indicate the binomial expansion of $\{D+\theta\}$, but a k-fold repetition of the operation Θ.

For $n>1$, the dimensions of the Jacobian can be reduced by unity by change of co-ordinates in such a fashion that the lines of flow lie on the surfaces (are characteristics of) $x'^n=$const. and $|\partial x'^i/\partial x^j|=1$. This can always be done. In these new co-ordinates, $u'^n=0$; $x'^n\equiv x^n$ and the Jacobian is a determinant of the same type, but of order $n-1$. The one change is that ordinary partial differentiation must be replaced by co-variant differentiation. As higher co-variant derivatives are not commutative in general, because of

$$u^i_{/j/k}-u^i_{/k/j}=R^i_{jkl}\,u^l \qquad \dots \quad (9)$$

we must, for the purpose of the deduction only, regard the manifold of the x^i as Euclidean. Then (8) will be in the invariantive form under change of co-ordinates, the theorem true for $n=m+1$ if true for $n=m$. But for $n=1$, the theorem is almost obvious, and very easily proved; hence, for all n. It must again be emphasized that (8) is to be regarded as a purely analytical formula, quite apart from the geometry that may afterwards be assigned to the manifold.

It is clear that $dJ/dt=\Theta J$. Therefore, we may state a corollary:

$$J=\sum_{k=0}^{\infty}\frac{t^k}{k!}\{D+p\Theta\}^k(1)=1+\sum_{k=1}^{\infty}\frac{t^k}{k!}\{D+p\Theta\}^{k-1}(p\Theta) \qquad \dots \quad (10)$$

A second theorem is very easily proved as before, or directly.

If ρ *transforms as in* (4), *then*

$$\bar{\rho}\,J=\rho+t\,(D\rho+\rho\Theta)+\dots=\sum_{k=0}^{\infty}\frac{t^k}{k!}\Theta^k\rho \qquad \dots \quad (11)$$

The best application is, of course, in hydrodynamics. The generators u^i are interpreted as the velocity-components of the fluid, and ρ the density, the entire movement being steady. For the conservation of mass of every portion of the fluid, we must have $\bar{\rho}J=\rho$. This leads to the result:

A necessary and sufficient condition for the continuity of fluid motion is that $\Theta\rho \equiv D\rho + \rho u^r,_r = 0$.

If this is satisfied initially, throughout the fluid, it will be satisfied everywhere for all time, and the motion which is continuous at the start in the sense of the conservation of mass, will always remain so continuous.

The case where t enters explicitly in the field of velocities, and the movement consequently depends on the time, can always be brought under the preceding by regarding t as an additional co-ordinate x°, with $u^\circ = 1$. The parameter of the group being labelled s, we have

$$\bar{t} = t + s \qquad \dots \quad (12)$$

$$D = \partial / \partial t + u^r \partial / \partial x^r$$

$$\Theta = u^r,_r$$

$$J = \partial(\bar{x}, \bar{t}) / \partial(x, t) = \partial(\bar{x}) / \partial(x) \qquad \text{as before.}$$

The previous expansion holds with the addition of a term $\partial / \partial t$ to D. The necessary and sufficient condition for continuity is

$$\frac{\partial\rho}{\partial t} + u^r \frac{\partial\rho}{\partial x^r} + \rho u^r,_r = 0 \qquad \dots \quad (13)$$

which must now be fulfilled everywhere and for all time.

The same problem for Riemannian spaces is very easily solved, and of some importance in the theory of relativity. If $\mu^2 = \pm \mid g_{ij} \mid \neq 0$ be the discriminant of the space, the conservation of mass comes to

$$\bar{\rho}\,\bar{\mu} J = \rho\mu \qquad \dots \quad (14)$$

From the preceding,

$$\bar{\rho}\,\bar{\mu}\,J = \sum_{k=0}^{\infty} \frac{t^k}{k!}\,\Theta^k(\rho\mu) \qquad \dots \quad (15)$$

Here, Θ is still to be computed with ordinary partial derivatives and therefore not in a form independent of the choice of co-ordinates.

However, ρ, being an absolute invariant, has the same partial and co-variant derivative: $\rho_{,i} = \rho_{/i}$. From the well-known properties of Christoffel symbols, we have

$$u^i_{/j} = u^i_{,j} + \Gamma^i_{jr} u^r$$

$$\mu_{,k} = \mu \Gamma^r_{rk} \quad \dots \quad (16)$$

Therefore,

$$\Theta \rho\mu = \{ \mu u^r \rho_{,r} + \rho u^r \mu_{,r} + \rho \mu u^r_{,r} \} = \mu \{ u^r \rho_{/r} + \rho u^r_{/r} \} \dots (17)$$

That is, the operation Θ performed upon μ times an absolute invariant, gives μ times another absolute invariant, as the operator represented by $\Theta' f = u^r f_{/r} + f u^r_{/r}$ is of the invariantive sort.

Consequently,

$$\bar{\rho}\,\bar{\mu}\,J = \mu \sum_{k=0}^{\infty} \frac{t^k}{k!} \Theta'^k \rho \quad \dots \quad (18)$$

The equation of continuity for Riemannian spaces can therefore be written in a form similar to the previous,

$$\Theta' \rho \equiv u^r \rho_{/r} + \rho u^r_{/r} = 0 \quad \dots \quad (19)$$

It must be noted that the direct substitution of co-variant for partial differentiation in computing $\partial \bar{x}^i / \partial x^j$ and the subsequent calculation of J would not only be not justified, but would not give the proper result, because of the non-commutability of co-variant derivatives shown in (9).

Notes and References :

The theorem on homogeneous functions, according to Cantor's *Vorlesungen uber die Geschichte d. Math.* appeared in Euler's book on the differential calculus in 1755. As it occupies so fundamental a position in many other sabjects, it has never been thought of as a theorem in continuous groups.

The second theorem appeared in the same year in two different publications (cf. Lamb : *Hydrodynamics*, VI ed., p. 2). The treatment by means of continuous groups usually goes as far as the infinitesimal transformation, the actual expansion of the Jacobian for the finite transformation being omitted. As a result, a certain amount of confusion on the subject is to be found even in standard text-books. For instance, Appell (*Traite de Mecanique Rationelle III*, 1928, pp. 323-4) deduces the formula $dJ/dt = \theta J$ in place of $dJ/dt = \Theta J$. A somewhat different point of view is represented in Levi-Civita's *Absolute Differential Calculus*, pp. 361-2. There, one finds a treatment of a special finite transformation $\bar{x}^i = x^i + tu^i$. As a matter of fact, if this is to represent the movement of a liquid from our point of view, with t as the time and u^i the velocity field, we must have $Du^i = 0$, which gives the special case of fluid motion wherein the pressure is caused by the external force only, the movement being rectilinear and with constant velocity on each streamline. But the Jacobian of such a transformation is precisely the same as for the transformation $\bar{x}^i = x^i + tu^i + \frac{1}{2} t^2 Du^i$ provided $(Du^i)_{,j} = 0$. Therefore, the condition that the Jacobian be identically equal to unity would represent, in general, the case of rectilinear but uniformly accelerated movement $(Du^i)_{,j} = 0$.

The proof of formula (19) is given in detail only to show to the advanced student that an immediate substitution of co-variant for ordinary partial differentiation has to be justified.

For an absolute invariant f, we have a formula analogous to (10), *i.e.*,

$$\bar{f} = f\{ 1 + t\phi + \frac{t^2}{2\,!}(D\phi + \phi^2) + \ldots \} = f \sum_{k=0}^{\infty} \frac{t^k}{k\,!} \Phi^k (1)$$

where $$\phi = \frac{Df}{f} \text{ and } \Phi = D + \phi.$$

ON THE EQUATION
$A_1^n + A_2^n + \dots + = A_r^n = B_1^n + \dots + B_r^n$, $(n=1, 2, 3, 4, 5, 6)$
AND ALLIED FORMS.*

BY ALFRED MOESSNER in *Nurnburg, Germany.*

I. Let $$[a, a_2, \dots]^n = [b_1, b_2, \dots]^n$$

mean $$a_1^n + a_2^n + \dots = b_1^n + b_2^n + \dots$$

(a) If $a + b = c$, then we have identically

$[sa - bx, sb + cx, -sc - ax]^n = [sb - ax, -sc - bx, sa + cx]^n$, $(n = 1,2,4)$ where s and x are arbitrary.

If the negative sign is not allowed, one can obtain from the above, relations of the form

(1) $[C_1, C_2, C_1 + C_2]^n = [D_1, D_2, D_1 + D_2]^n$, $(n = 2, 4)$.

For example, if we set $a = 2$, $b = 1$, $c = 3$, $s = 1$, $x = 3$, then we get

$$[-1, 10, -9]^n = [-5, -6, 11^n], \qquad (n = 1, 2, 4).$$

Hence $$[1, 10, 9]^n = [5, 6, 11]^n, \qquad (n = 2, 4).$$

Again, if (1) is true, then

$$C_1^2 + C_2^2 + C_1.C_2 = D_1^2 + D_2^2 + D_1.D_2$$

and further it follows that

$$2(C_1^2 + C_2^2 + C_1.C_2)^2 = 2(D_1^2 + D_2^2 + D_1.D_2)^2,$$
$$= [C_1, C_2, C_1 + C_2]^4 = [D_1, D_2, D_1 + D_2]^4$$

Example. $[2, 11, 13]^n = [7, 7, 14]^n$, $(n = 2, 4)$.

Hence, $$2^2 + 11^2 + 2.11 = 7^2 + 7^2 + 7.7,$$

and $2(2^2+11^2+2.11)^2 = 2(7^2+7^2+7.7)^2 = [2, 11, 13]^4 = [7, 7, 14]^4$.

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* [Translated from the original German—ED.].

(β) One obtains also relations of the form (1) above, by setting

$$C_1 = 4f - g; \qquad C_2 = 3g - 5f \qquad -(C_1 + C_2) = f - 2g$$
$$D_1 = 2g - 5f; \qquad D_2 = f + g \qquad -(D_1 + D_2) = 4f - 3g.$$

Thus $f = 3$, $g = 8$ give $[4, 9, 4 + 9]^n = [1, 11, 1 + 11]^n$ $(n = 2, 4)$.

(γ) If in (α) we choose a, b, c, s, x so that one of the terms becomes zero, then we arrive at identities of the form

$$L^n + L^n = M^n + N^n + Q^n \qquad (n = 2, 4).$$

We can obtain such identities from simpler formulae:

$$L = f^2 + fg + g^2; \quad M = f^2 - g^2; \quad N = 2fg + g^2; \quad Q = M + N.$$

Here we have to choose f and g so that $f > g$. Thus, $f = 5$, $g = 3$, give

$$49^n + 49^n = 16^n + 39^n + (16 + 39)^n, \qquad (n = 2, 4).$$

II. If the relation

$$[C_1, C_2, C_3]^n = [D_1, D_2, D_n]^n \qquad (n = 2, 4)$$

holds then we have

$$[T - C_3, T - C_2, T - C_1, T + C_1, T + C_2, T + C_3]^n$$
$$= [T - D_3, T - D_2, T - D_2, T + D_1, T + D_2, T + D_3]^n,$$

$n = 1, 2, 3, 4, 5$, where T is arbitrary.

In this way, we arrive at identities of the form

$$\sum_{r=1}^{6} E_r^n = \sum_{r=1}^{6} F_r^n, \qquad (n = 1, 2, 3, 4, 5).$$

Example. $C_1 = 4$, $C_2 = 15$, $C_3 = 19$, $D_1 = 11$, $D_2 = 11$, $D_3 = 20$, $T = 21$, gives

$$[2, 6, 17, 25, 36, 40]^n = [1, 10, 12, 30, 32, 41]^n, \quad (n = 1, 2, 3, 4, 5).$$

III. (α) From the formula in II, we obtain the relation

$$2L^n = M^n + N^n + Q^n, \qquad (n = 2, 4);$$

or

$$[0, L, L]^n = [M, N, Q]^n, \qquad (n = 2, 4);$$

and identities of the form

$$2 \sum_{r=1}^{3} S_r^n = \sum_{r=1}^{6} V_r^n, \qquad (n = 1, 2, 3, 4, 5).$$

Example. From $2.13^n = 7^n + 8^n + 15^n$, $(n = 2, 4)$,
if we put $T = 16$, we obtain the identity

$$2\,[3, 16, 29]^n = [1, 8, 9, 23, 24, 31]^n, \qquad (n = 1, 2, 3, 4, 5).$$

(β) Given the identity $2.L^n = [M, N, Q]^n$, $(n = 2, 4)$,

the following relations hold good:

$2\,L^{rn} = [L^{r-1}M, L^{r-1}N, L^{r-1}Q]^n$, $n = 2, 4$ and $r = 2, 3, 4, 5$, etc.

Example. $2.7^n = [3, 5, 8]^n$, $(n = 2, 4)$.

Hence $2.7^{2n} = [21, 35, 56]^n$, $(n = 2, 4)$;

$2.7^{3n} = [147, 245, 392]^n$, $(n = 2, 4)$.

IV. From $[E_1, E_2, \ldots E_6]^n = [F_1, F_2, \ldots F_6]^n$ $(n = 1, 2, 3, 4, 5)$.

it follows that

$$\sum_{r=1}^{6} \{ E_r^n + (F_r + k)^n \} = \sum_{r=1}^{6} \{ F_r^n + (E_r + k)^n \}, \quad (n = 1, 2, 3, 4, 5, 6).$$

where k is any number. Thus we get identities of the form

$$[A_1, A_2, \ldots A_z]^n = [B_1, B_2, \ldots B_z]^n, \qquad (n = 1, 2, 3, 4, 5, 6).$$

Example.

$$[2, 3, 11, 13, 21, 22]^n = [1, 6, 7, 17, 18, 23]^n, \qquad (n = 1, 2, 3, 4, 5).$$

If we put $k = 5$, then we obtain the relation

$$[2, 3, 11, 11, 12, 13, 21, 22, 22, 28]^n = [1, 7, 7, 8, 16, 17, 18, 18, 26, 27]^n,$$
$$(n = 1, 2, 3, 4, 5, 6).$$

If, in the identity $2\,[S_1, S_2, S_3]^n = [V_1, V_2, \ldots V_6]^n$ $(n = 1, 2, 3, 4, 5)$.

we put $k = S_2 - S_1 = S_3 - S_2$, then by the above method we obtain the relation

$$[G_1, G_2, \ldots G_8]^n = [H_1, H_2 \ldots H_8]^n, \qquad (n = 1, 2, 3, 4, 5, 6).$$

Example. $2\,[2, 9, 16]^n = [1, 4, 6, 12, 14, 17]^n$, $(n = 1, 2, 3, 4, 5)$.

If we put $k = 9 - 2 = 16 - 9 = 7$, then we get

$$[2, 2, 8, 11, 13, 19, 21, 24]^n = [1, 4, 6, 12, 14, 17, 23, 23]^n,$$
$$(n = 1, 2, 3, 4, 5, 6).$$

REMARKS ON THE PREVIOUS PAPER OF DR. MOESSNER.

BY S. S. PILLAI, *Annamalainagar*.

I. In the previous note, Dr. Moessner observes that, if

$$\sum_{r=1}^{6} E_r^n = \sum_{r=1}^{6} F_r^n, \qquad n = 1, 2, 3, 4, 5,$$

then it follows that

$$\sum_{r=1}^{6} \{ E_r^n + (F_r + k)^n \} = \sum_{r=1}^{6} \{ F_r^n + (E_r + k)^n \}, \quad n = 1, 2, 3, 4, 5, 6.$$

This implies the general result, namely, if

$$\sum a_t^n = \sum b_t^n, \qquad n = 1, 2, 3, \ldots, l, \qquad \ldots \quad (1)$$

$$\text{then } \sum a_t^n + \sum (b_t + k)^n = \sum b_t^n + (a_t + k)^n, \quad n = 1, 2, \ldots, l+1, \qquad \ldots \quad (2)$$

where the summation is for t from 1 to r.

This gives the interesting result that it is possible to find two sets of positive integers such that the sum, the sum of squares, the sum of cubes, ... , the sum of say, the 1000th powers of numbers in the first set are equal to the corresponding sums of the second set. For example,

$$(2 + {}_{1000}C_1)\, 2^n + ({}_{1000}C_1 + 2.{}_{1000}C_2 + {}_{1004}C_3)4^n$$
$$+ ({}_{1000}C_3 + 2{}_{1000}C_4 + {}_{1000}C_5)6^n.$$
$$+ ({}_{1000}C_5 + 2{}_{100c}C_6 + {}_{1000}C_7)8^n + \ldots \ldots \ldots$$
$$= 1^n + (1 + 2).{}_{1000}C_1 + {}_{1000}C_2)3^n$$
$$+ ({}_{1000}C_2 + 2.{}_{1000}C_3 + {}_{1000}C_4)5^n + \ldots \ldots \ldots$$

for $n = 1, 2, 3, \ldots, 1000$ and 1001.

We have got only 2^{1001} terms on each side !

Let (1) be true. In (2), put $k = 1$. Then we get

$$\sum a_t^n + \sum (b_t + 1)^n = \sum b_t^n + (a_t + 1)^n, \qquad n = 1, 2, \ldots, l+1.$$

Apply the above result again to this by taking 1 for k.

$$\sum a_t^n + 2\sum (b_t + 1)^n + \sum (a_t + 2)^n = \sum b_t^n + 2\sum (a_t + 1)^n + \sum (b_t + 2)^n,$$
$$n = 1, 2, \ldots, l+2.$$

Repeating this process m times, we obtain

$$\Sigma\, a_t^n + {}_mC_1\, \Sigma\, (b_t+1)^n + {}_mC_2\, \Sigma\, (a_t+2)^n + {}_mC_3\, \Sigma\, (b_t+3)^n + \dots\dots$$
$$= \Sigma\, b_t^n + {}_mC_1\, \Sigma\, (a_t+1)^n + {}_mC_2\, \Sigma\, (b_t+2)^n + \dots \; (n = 1, \dots, l+m). \quad \dots \quad (3)$$

Now $1 + 4 = 2 + 3$. So

$$1^n + 4^n + (2+1)^n + (3+1)^n = 2^n + 3^n + (1+1)^n + (4+1)^n$$

i.e., $$1^n + 2.4^n = 2.2^n + 5^n. \qquad (n = 1, 2).$$

So $1^n + 2.4^n + 2.(2+1)^n + (5+1)^n = 2.2^n + 5^n + (1+1)^n + 2.(4+1)^n$

i.e., $$1^n + 2.3^n + 2.4^n + 6^n = 3.2^n + 3.5^n, \qquad (n = 1, 2, 3)$$

and so on.

If we start from $2+2 = 1+3$, we get the example given in I.

III. Let N (k) be the smallest N such that

$$\sum_{t=1}^{N} x_t^n = \sum_{t=1}^{N} y_t^n, \qquad (n = 1, 2, 3, \dots, k)$$

(where the x_t's are not identical with y_t's) has got infinity of primitive solutions. Then what can we say about N (k)?

$$N(1) = 2; \qquad N(2) = 3; \qquad N(5) = 6, \text{ etc.}$$

If $$\sum_{r=1}^{k} a_r^n = \sum_{r=1}^{k} b_r^n, \qquad (n = 1, 2, \dots, k),$$

then, from the theory of equations, it is easy to deduce that

$$a_r = b_r, \qquad r = 1, 2, \dots, k.^*$$

Hence if follows that $\quad N(k) \geqslant k + 1.$

Since $x + y = u + v$ has got infinity of primitive solutions, by putting $\quad a_1 = x,\ a_2 = y,\ b_1 = u,\ b_2 = v,\ l = 1,$

we get from (3) that $N(k) \leqslant 2^k$.

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* It follows that corresponding symmetric functions of a_r and b_r are equal. Hence a_r and b_r are the roots of the same equation of kth degree.

Thus we obtain $k + 1 \leqslant N(k) \leqslant 2^k$.

But, if we start from $1 + 4 = 2 + 3$, then from (3) we get

$$1^n + 4^n + {}_mC_1(3^n + 4^n) + {}_mC_2(3^n + 6^n) + {}_mC_3(5^n + 6^n) + \dots\dots$$
$$= 2^n + 3^n + {}_mC_1(2^n + 5^n) + {}_mC_2(4^n + 5^n) + \dots\dots,$$

for $n = 1, 2, 3, \dots, m$.

Let T be the number of terms on each side when we cancel out common terms. Then it can be easily verified that $T < 4\ {}_mC_\mu$ where μ is the integral part of $m/2$.

So we get that

$$\sum_{t=1}^{r} a_t^n = \sum_{t=1}^{r} b_t^n, \qquad (n = 1, 2, \dots, m) \quad \dots \quad (4)$$

has got at least one solution when $r = T$. Hence from the general remark in I, it follows that (4) has got infinity of primitive solutions when $r = 2T$.

So $N(m) \leqslant 2T = O({}_mC_\mu)$.

Hence $N(k) = O(2^k/\sqrt{k})$

But I am not able to improve the result further.

GLEANINGS.

29. "Time was when a theorem could constitute a considerable contribution to Mathematical Science. But now theorems are turned out wholesale. A single treatise will contain hundreds of them. Nowadays methods alone can arrest attention strongly; and these are coming in such flocks that the next step will surely be to find *methods of discovering methods*. This can only come from the theory of the method of discovery. In order to cover every possibility, this could be founded on the general doctrine of methods of attaining purposes, in general; and this, in turn, should spring from the still more general doctrine of the nature of teleological action, in general."

C. S. PEIRCE.

30. There is no necessity for supposing that the process of thought as it takes place in the mind, is always cut up into distinct arguments.

C. S. PEIRCE.

TETRAHEDRAL POLES AND POLAR PLANES.

BY NATHAN ALTSHILLER COURT,
University of Oklahoma, U. S. A.

1. Given two skew lines a, b, and a point P, the line is drawn through P meeting a and b in the points A, B. The harmonic conjugate Q of P with respect to A, B, is said to be the "harmonic conjugate of P with respect to a, b."

2. *If a point describes a plane, the harmonic conjugate of this point with respect to two skew lines also describes a plane.*

Let C, D be the traces of the given lines a, b in the given plane, and let P be *any* point of this plane. The harmonic conjugate Q of P with respect to a, b lies in the plane CDQ which is harmonically separated from the given plane CDP by the two fixed planes CD-a, CD-b, hence the plane CDQ is fixed.

3. The plane CDQ may be called the "harmonic congugate" of the plane CDP with respect to the two given skew lines a, b. It is evident that the relation between the two planes CDP, CDQ is reciprocal.

4. *If a point describes a straight line, the harmonic conjugate of this point with respect to two skew lines also describes a straight line.*

Indeed, the harmonic conjugates Q, Q$'$, Q$''$, ... of the points P, P$'$, P$''$, ... of the given line u with respect to the two given skew lines a, b must lie in the harmonic conjugate plane, with respect to a, b of any plane passing through u, hence they lie on a line u'.

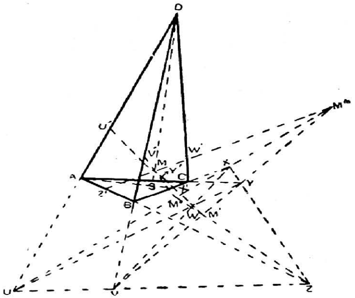
5. The line u' may be called the "harmonic conjugate" of u with respect to a, b.

It is clear that the relation between u and u' is reciprocal, and, furthermore, that the lines a, b are harmonic conjugates with respect to u, u'.

6. *If a line meets two opposite edges of a tetrahedron, its two harmonic conjugates with respect to the other two pairs of opposite edges coincide.*

Let U$'$, X$'$ be the traces of the given line U$'$X$'$ on the opposite edges DA, BC of the tetrahedron DABC. The harmonic conjugate

of U′X′ with respect to AB, CD is determined by the harmonic conjugates U, X of U′, X′ with respect to the pairs of points A, D ; B, C. But the harmonic conjugate of the line U′, X′ with respect to the pair of lines AC, BD also contains the points U, X, as is readily seen by considering the transversals AU′D, BX′C, hence the proposition.



7. For the sake of brevity, we shall say that UX is the "harmonic conjugate of U′X′ with respect to the tetrahedron DABC." It is clear that the relation between U′X′ and UX is reciprocal.

8. The plane determined by the three harmonic conjugates of a given point with respect to the three pairs of opposite edges of a tetrahedron is referred to as the "harmonic plane," or the "polar plane," or the "tetrahedral polar plane" of the given point with respect to the tetrahedron.

9. *The harmonic conjugates, with respect to a tetrahedron, of the three lines passing through a given point and meeting the three pairs of opposite edges of the tetrahedron lie in the harmonic plane of the point with respect to the tetrahedron.*

Let d, e, f be three lines passing through the given point O and meeting the three pairs of opposite edges a, a' ; b, b' ; c, c', of the given tetrahedron (T). The harmonic conjugate d' of d with respect to (T) passes through the harmonic conjugates of O with respect to b, b' ; c, c'. Hence d' lies in the harmonic plane of O with respect to (T). Similarly for e, f.

10. The point determined by the three harmonic conjugates of a given plane with respect to the three pairs of opposite edges of a tetrahedron is referred to as the "harmonic pole," or the "tetrahedral pole," or the "pole" of the given plane with respect to the tetrahedron.

11. *The harmonic conjugates, with respect to a tetrahedron, of three co-planar lines meeting, respectively, the three pairs of opposite edges of a tetrahedron, pass through the harmonic pole, with respect to the tetrahedron, of the plane containing the three given lines.*

Let p, q, r be the three given lines lying in the plane σ and meeting, respectively, the three pairs of opposite edges a, a' ; b, b' ; c, c' of the tetrahedron (T).

The harmonic conjugate p' of p with respect to (T) lies in the two harmonic conjugate planes of σ with respect to b, b' ; c, c'. Hence p' passes through the pole of σ with respect to (T). Similarly for q and r.

12. *If μ is the polar plane of the point M with respect to a tetrahedron, then M is the pole of the plane μ with respect to this tetrahedron.*

Let u', v', w' be the three lines passing through M and meeting the three pairs of opposite edges of the tetrahedron (T). The harmonic conjugates u, v, w of u', v', w' with respect to (T) lie in the plane μ (9).

On the other hand the harmonic conjugates of u, v, w with respect to (T) pass through the pole of μ with respect to (T) (11). Now these harmonic conjugates coincide with u', v', w' for the relation between conjugate lines is reciprocal (7), hence the pole of μ with respect to (T) coincides with M.

13. If U′, X′ ; U, X are the traces of the lines u', u (12) on the edges DA, BC of (T) ≡ DABC, the two groups of points (DA, UU′), (BC, XX′) are harmonic (4). Similarly for the traces V′, Y′ ; V, Y of v', v on the edges DB, CA and again for the traces W′, Z′ ; W, Z of w', w on DC, AB. Hence the following table of groups of harmonic points

(A)	(DA, UU′)	(DB, VV′)	(DC, WW′)
	(BC, XX′)	(CA, YY′)	(AB, ZZ′)

14. *The trilinear pole, with respect to a face of a given tetrahedron DABC, of the line of intersection of this face with a given plane*

μ is collinear with the vertex opposite the face considered and the pole of μ with respect to DABC.

The pole M of μ lies on the lines U'X', V'Y', W'Z' (12, 13), hence the trace of DM in ABC is the point S common to the lines AX', BY', CZ'. Now the trace of μ in the plane ABC contains the points X, Y, Z, hence S is the trilinear pole of XYZ.

15. *The trilinear polar, with respect to a face of a given tetrahedron DABC, of the trace in this face of the line joining the opposite vertex to a given point M lies in the polar plane of M with respect to DABC.*

The trace S of DM in ABC is common to the three lines AX', BY', CZ' (14), hence the trilinear polar of S with respect to ABC is determined by the points X, Y, Z (13), and these three points lie in the polar plane μ of M with respect to DABC.

16. The harmonic conjugates M', M'', M''', of M (12) with respect to the pairs of points U', X'; V', Y'; W', Z' (13) determine the tetrahedral polar plane of M with respect to the tetrahedron DABC (8), so that we have (13)

$$\mu \equiv M'M''M''' \equiv UVWXYZ.$$

The points M'', M''' are the traces on u = UX of the lines $v' \equiv$ V'Y', $v \equiv$ VY. Now v', v are harmonic conjugates with respect to DA, BC, hence M'', M''' are harmonically separated by U, X. Similarly for the lines $v \equiv$ VY, $w \equiv$ WZ We arrive thus at the following groups of harmonic points

(B)	(MM', U'X'),	(MM'', V'Y'),	(MM''', W'Z')
	(M''M''', UX),	(M'M''', VY),	(M'M'', WZ)

17. If the vertices of a tetrahedron (P) are the harmonic poles of the opposite faces of (P) with respect to a second tetrahedron (Q), then (P) is said to be "self-polar," or "harmonic" with respect to (Q).

18. *A point and its three harmonic conjugates with respect to the three opposite edges of a given tetrahedron are four vertices of a tetrahedron harmonic to the given tetrahedron.*

The table (B) (16) shows that through M' pass three lines M'U'X', M'VY, M'WZ, and these meet the three pairs of opposite edges of DABC in the three pairs of points U', X'; V, Y; W, Z; moreover

(B) shows that the harmonic conjugates of M′ with respect to these three pairs of points are M, M‴, M″ respectively. Consequently MM″M‴[1] is the harmonic plane of M′ with respect to DABC. Similarly for the points M″, M‴[1]. Thus MM′M″M‴[1] is harmonic with respect to DABC.

19. *If the tetrahedron (P) is self-polar with respect to the tetrahedron (Q), then (Q) is self-polar with respect to (P).*

Given the tetrahedron (P) = DABC and the point M (18) we constructed the tetrahedron (Q) = M′M″M‴[1]M self-polar with respect to (P). It follows from this construction that M is the vertex of one and only one such tetrahedron. Hence in order to prove the present proposition, it is sufficient to show that ABCD is harmonic with respect to MM′M″M‴.

From the harmonic group (DA, UU′) of table (A) (13) and the two harmonic groups (MM′, U′X′), (M″M‴, UX) of table (B) it follows that the line DUU′ meets the opposite edges MM′, M″M‴[1] of the tetrahedron MM′M″M‴ in the points U′, U and that A is the harmonic conjugate of D with respect to U′, U. Considering the lines DVV′, DWW′, we may show that the plane ABC is the polar plane of D with respect to the tetrahedron MM′M″M‴.

A similar reasoning may be applied to each of the points A, B, C. Hence the proposition.

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31. Women screamed and hundreds of penitents walked barefooted in the streets of the towns throughout Portugal last night in fear of a cataclysm when they saw the sky filled with millions of shooting stars. The unique display was due to the *Earth's orbit passing through the orion nebula.*

Extract from an English newspaper referring to the meteor shower of 9th October 1933.

(*Italics ours*.—ED.)

32. "In the curse of my law reading I constantly came upon the word "Demonstrate." I thought at first that I understood its meaning, but soon became satisfied that I did not. I consulted all the dictionaries and works of reference I could find * * At last I said. 'Lincoln, you can never make a lawyer if you do not understand what DEMONSTRATE means,' and I left my situation in Springfield, went home to my father's house and stayed there till I could give any proposition in the six books of Euclid at sight. I then found out what demonstrate means and went back to my law studies."

ABRAHAM LINCOLN.

NOTES AND DISCUSSIONS.

The Editor welcomes for publication under this heading, brief discussions of interesting problems, critical comments, and suggestions likely to be helpful in the class-room.

On a certain Theorem in Derangements.

In this note, I give a new method for determining in how many ways an ordered aggregate of n letters may be deranged so that no one out of r assigned letters shall occupy its original position.

Let $a_1, a_2, \dots a_n$ be the n letters in order. There is no loss of generality in supposing that the r assigned letters are $a_1, a_2, \dots a_r$. Let

$$\Sigma = a_1 + a_2 + \dots + a_n$$

$$\sigma_1 = a_1 + a_2 + \dots + a_r$$

$$\dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots$$

σ_s = sum of all the s-ary combinations of the r letters $a_1, a_2 \dots a_r$.

$$\omega = a_1 \, a_2 \, \dots \, a_n .$$

In any derangement of letters fulfilling the conditions of the problem, the first place can be occupied by any one of the letters $a_2, a_3 \dots a_n$.

That is, all the possibilities of filling the first place are found among the terms in $a_2 + a_3 + \dots + a_n$ which is equal to $(\Sigma - a_1)$.

Similarly the independent possibilities for the 2nd, 3rd, ... rth places are given by $(\Sigma - a_2)$, $(\Sigma - a_3)$, ... $(\Sigma - a_r)$ and for each of the remaining $(n - r)$ places it is Σ.

Hence the totality of all derangements of the letters, fulfilling the conditions of the problem, is in (1, 1) correspondence with the various ways of forming the term $\omega \equiv a_1, a_2 \dots a_n$ in the product P,

where $\qquad P \equiv (\Sigma - a_1)(\Sigma - a_2) \dots (\Sigma - a_r)\, \Sigma^{n-r}$

$$= \Sigma^n - \sigma_1 \Sigma^{n-1} + \dots + (-)^s \sigma_s . \Sigma^{n-s} + \dots + (-)^r \sigma_r \Sigma^{n-r} \dots \qquad (1)$$

so that the required number is the co-efficient of ω in P.

Now σ_s is the sum of all the ${}_rC_s$ s-ary combinations of the r letters $a_1, a_2, \ldots a_r$, and each term gives ω once, when multiplied by the product of the remaining $(n-s)$ letters, and the co-efficient of this last product in Σ^{n-s} is clearly $(n-s)!$ Hence the co-efficient of ω in $\sigma_s . \Sigma^{n-s}$ is ${}_rC_s . (n-s)!$

Therefore the co-efficient of ω in P is

$$n! - {}_rC_1 (n-1)! + {}_rC_2 (n-2)! - \ldots + (-)^r (n-r)!$$

which is the number of derangements.

Taking $r = n$, we get the number of derangements of n letters in which no one letter occupies its original position, namely,

$$n! - {}_nC_1 (n-1)! + {}_nC_2 (n-2)! - \ldots + (-)^n . 1.$$

The theorem was first proved by Euler making use of a recurrence formula, and a proof on the same lines is given in Chrystal's *Algebra*, Part II.

R. S. VAIDISWARAN.

On Powers of Numbers.

Following the notation of a preceding paper,* I here prove

THEOREM 1. $(1) \overset{5}{=} (9)$ *i.o.*

THEOREM 2. $(2) \overset{5}{=} (8)$ *i.o.*

THEOREM 3. $(3) \overset{5}{=} (6)$ *i.o.*

We have†

$$(x+y+z)^5 = (y+z-x)^5 + (z+x-y)^5 + (x+y-z)^5 + 80xyz(x^2+y^2+z^2) \ldots (1)$$

Putting $y = 25n^5, \quad z = 50n^5$ and $x = 4k^5$

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* K. Subba Rao "On Powers of Numbers," *The Math. Student*, Vol. II, No. 1 (1934), pp. 6–10.

† This identity is taken from S. Sastry's paper to appear in *Jour. Lond. Math. Soc.*

in (1), we have

$$(4k^5 + 75n^5)^5 = (75n^5 - 4k^5)^5 + (4k^5 + 25n^5)^5 + (4k^5 - 25n^5)^5 + 2(20n^2k^3)^5 + 4(50n^4k)^5 \quad \ldots \quad (2)$$

If in (2), $25n^5 < 4k^5 < 75n^5$,

we have $(1) \overset{5}{=} (9)$ *i.o.*

On the other hand, if in (2), $4k^5 < 25n^5$

we have $(2) \overset{5}{=} (8)$ *i.o.*

Hence Theorems 1 and 2 are proved.

Again, we have

$$(10t^5 + 1)^5 - (10t^5 - 1)^5 = (10t^4)^5 + 2000\,(t^2)^5 + 2$$
$$= (10t^4)^5 + (5t^2)^5 - 1125\,(t^2)^5 + 1^5 + 1^5 \quad \ldots \quad (3)$$

Now $1125 \times 5^5 = 20^5 + 12^5 + 9^5 + 6^5 - 2^5.$

Substituting this in (3) and rearranging, we have

$$(50t^5 + 5)^5 + (20t^2)^5 + (12t^2)^5 + (9t^2)^5 + (6t^2)^5$$
$$= (50t^5 - 5)^5 + (50t^4)^5 + (25t^2)^5 + (2t^2)^5 + 5^5 + 5^5$$

which proves Theorem 3.

K. SUBBA RAO.

On the tests for the divisibility of an integer by another.

Tests for the divisibility of an integer by 3, 4, 9, 11 are well-known. In this note, I discuss the rationale of these and other similar tests.

Since we can have only n different remainders when integers are divided by n, at least two of the numbers in the set $1, 10, 10^2 \ldots, 10^n$ should leave the same remainders. Let the first such pair be 10^a and 10^{a+k}. Then $10^{a+k} - 10^a$, that is, $10^a(10^k - 1)$ is a multiple of n. Now $10^{a+mk+t} - 10^{a+t} = 10^t.10^a.(10^{mk} - 1)$ and $10^{mk} - 1$ is a multiple of $10^k - 1$. So $10^{a+mk+t} - 10^{a+t}$ is a multiple of n. Hence, if $r_1, r_2, r_3 \ldots$ are the remainders when $1, 10, 10^2, 10^3 \ldots$ are divided by n, it follows that $r_{a+mk+t} = r_{a+t}$. That is,

$$r_{a+t} = r_{a+t+k} = r_{a+t+2k} = \ldots, \; (t = 0, 1, \ldots, k-1),$$

Consequently, the remainders, when 1, 10, 10^2 ... are divided by n, form the periodic sequence

$$1,\ r_1,\ r_2,\ \dots r_{a-1},\ \overset{*}{r_a},\ r_{a+1},\ \dots,\ \overset{*}{r}_{a+k-1} \qquad \dots \quad (1)$$

where $1, \dots, r_{a-1}$, is the non-periodic part and $r_a, r_{a+1}, \dots r_{a-k-1}$ are repeated in the same order.

Let $R(x/y)$ be the remainder when x is divided by y, and let $N = d_m d_{m-1} \dots d_1 d_0$ in the usual decimal notations. Then

$$\begin{aligned} R(N/n) &= R\{(d_0 + 10d_1 + \dots + 10^m d_m)/n\} \\ &= R\{(d_0 + r_1 d_1 + \dots + r_m d_m)/n\} \\ &= R\{[d_0 + r_1 d_1 + \dots + r_{a-1} d_{a-1} \\ &\quad + \sum_{t=0}^{k-1} r_{a+t}(d_{a+t} + d_{a+t+k} + d_{a+t+2k} + \dots)]/n\} \end{aligned}$$

from (1) and (2).

(1) $n = 3,\ 1 = r_0 = r_1 = r_2 = \dots \dots$
$R(N/3) = R\{(d_0 + d_1 + \dots + d_m)/3\}$.

(2) Similarly for 9.

(3) $n = 8,\ r_0 = 1,\ r_1 = 2,\ r_2 = 4,\ r_3 = r_4 = \dots \dots = 0.$
So $R(N/8) = R\{(d_0 + 2d_1 + 4d_2)/8\} = R\{(d_2 d_1 d_0)/8\}$.

(4) $n = 12 \quad r_0 = 1,\ r_1 = 10,\ r_2 = r_3 = r_4 = \dots \dots = 4.$
So $R(N/12) = R\{[d_0 + 10d_1 + 4(d_2 + d_3 + \dots + d_m)]/12\}$.

(5) $n = 11, \quad r_0 = r_2 = r_4, \dots = 1;\ r_1 = r_3 = \dots \dots = 10.$

So $R(N/11) = R\{[d_0 + d_2 + d_4 + \dots + 10(d_1 + d_3 + d_5 + \dots)]/11\}$.

So if N is divisible by 11, then

$$0 = R(N/11) = R\{[d_0 + d_2 + d_4 + \dots + (11-1)(d_1 + d_3 + \dots +)]/11\}$$
$$= R\{[(d_0 + d_2 + \dots) - (d_1 + d_3 + \dots)]/11\}.$$

Hence $(d_0 + d_2 + \dots) - (d_1 + d_3 + \dots \dots)$ is divisible by 11.

(6) $n = 125,\ r_0 = 1,\ r_2 = 10,\ r_2 = 100,\ r_3 = r_4 = \dots = 10.$
So $R\{N/125\} = R\{(d_2 d_1 d_4)/125\}$, etc.

A. M. DESMPANDE.

An operational solution of $\frac{dy}{dx} + Py = Q$ where P and Q are functons of x only.

1. If P and Q are functions of x, an operational solution of $\frac{dx}{dy} + Py = Q$ or $(D + P)\, y = Q$, is given by

$$y = \frac{1}{D + P}\, Q \qquad \dots \quad (1)$$

or

$$y = \frac{1}{D}\left\{1 - \left(\frac{P}{D}\right) + \left(\frac{P}{D}\right)^2 - + \dots\right\} \qquad \dots \quad (2)$$

where

$$\frac{P}{D}\, Q = PD^{-1}Q = P \int Q\, dx.$$

$$\left(\frac{P}{D}\right)^r Q = \left(\frac{P}{D}\right)\left(\frac{P}{D}\right)^{r-1} Q \qquad (r \geqslant 2).$$

Example : If $\frac{dy}{dx} + xy = x$

then $y = \frac{1}{D}\left\{1 - \frac{x}{D} + \left(\frac{x}{D}\right)^2 - + \dots\right\} x = \frac{x^2}{2} - \frac{x^4}{8} + - \dots$

is a solution.

It is easy to show that (3) *is*, under certain convergence conditions, *actually a solution of the given differential equation.*

From (2) we also get

$$y = \frac{1}{P}\left\{1 - \frac{D}{P} + \left(\frac{D}{P^2}\right) - + \dots\right\} Q \qquad \dots \quad (3)$$

where

$$\frac{D}{P}\, Q = D\left(\frac{Q}{P}\right) = \frac{d}{dx}\left(\frac{Q}{P}\right)$$

$$\left(\frac{D}{P}\right)^r Q = \frac{D}{P} \cdot \left(\frac{D}{P}\right)^{r-1} Q \qquad (r \geqslant 2).$$

Example : If $\frac{dy}{dx} + xy = x$, then $y = 1$ is a solution.

It is important to observe that *in* (2) *we do not, in general, get a solution if we interpret*

$$\frac{P}{D}\, Q \quad as \quad \frac{1}{D}(PQ).$$

Similarly *in* (3) *we cannot interpret* $\frac{D}{P}$ Q *as meaning*

$$\frac{1}{P} \cdot D\,(Q) = \frac{1}{P}\,\frac{dQ}{dx}.$$

2. An operational solution of

$$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R$$

where P, Q, R are functions of x only, is given by

$$y = D^{-2}\left(1 + \frac{P}{D} + \frac{Q}{D^2}\right)^{-1} R$$

$$= D^{-2}\Big[1 - \left(\frac{P}{D} + \frac{Q}{D^2}\right)$$

$$+ \left\{\left(\frac{P}{D}\right)^2 + \left(\frac{Q}{D^3}\right)^2 + \left(\frac{P}{D}\right)\left(\frac{Q}{D^2}\right) + \left(\frac{Q}{D^3}\right)\left(\frac{P}{D}\right)\right\}$$

$$-\left\{\left(\frac{P}{D}\right)^3 + \left(\frac{Q}{D^3}\right)^3 + \left(\frac{Q}{D^2}\right)^2\left(\frac{P}{D}\right) + \left(\frac{Q}{D^2}\right)\left(\frac{P}{D}\right)\left(\frac{Q}{D^2}\right)\right.$$

$$+ \left(\frac{P}{D}\right)\left(\frac{Q}{D^2}\right)^2 + \left(\frac{P}{D}\right)^2\left(\frac{Q}{D^2}\right) + \left(\frac{P}{D}\right)\left(\frac{Q}{D^2}\right)\left(\frac{P}{D}\right)$$

$$\left. + \left(\frac{Q}{D^2}\right)\left(\frac{P}{D}\right)^2\right\}$$

$$+ - \ldots\Big]\, R$$

where $\frac{P}{D}\,R = PD^{-1}R, \quad \frac{Q}{D^4}\,R = QD^{-2}R$, etc., etc.

The symbols used are not commutative and any other interpretation $\left[e.g., \frac{P}{D}R = \frac{1}{D}(PR)\right]$ *would, in general, lead to false results.*

It is evident that we can proceed similarly with equations of higher order than the second.

I. CHOWLA.

SOLUTIONS TO QUESTIONS.

Question 1617.

(R. VAIDYANATHASWAMY) :—The θ normal at a point P of a conic is defined to be the line obtained by rotating the tangent at P through an angle θ in the positive direction about P. A θ-normal which passes through the centre of the conic is called a θ-diameter.

If the sides of a triangle inscribed in a conic are respectively parallel to the θ_1, θ_2 and θ_3 diameters of the conic, show that the θ normals to the conic at its vertices are concurrent, θ being given by

$$\cot\theta = \cot\theta_1 + \cot\theta_2 + \cot\theta_3.$$

Solution by S. Narayanan.

Let the eccentric angles of the vertices of the triangle inscribed in the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ be α, β, γ. Since the θ-normals at the vertices are concurrent, $2ab\cot\theta = c^2\,\Sigma\sin(\beta+\gamma)$ ... (1) (*Vide* Casey's *Analy. Geometry*, page 539, Question 116).

The side given by

$$\frac{x}{a}\cos\frac{\beta+\gamma}{2} + \frac{y}{b}\sin\frac{\beta+\gamma}{2} = \cos\frac{\beta-\gamma}{2}$$

is parallel to the θ_1 diameter of the conic. The θ_1 diameter at any point 'ϕ' of the ellipse is $y = \frac{bx}{a}\tan\phi$ where

$$\cot\theta_1 = -\frac{a^2-b^2}{ab}\sin\phi\cos\phi$$

By the condition of parallelism, $\tan\phi = -\cot\frac{\beta+\gamma}{2}$.

$$\therefore \qquad \cot\theta_1 = \frac{a^2-b^2}{2ab}\sin(\beta+\gamma).$$

$$\Sigma\cot\theta_1 = \frac{a^2-b^2}{2ab}\,\Sigma\sin(\beta+\gamma) = \cot\theta \text{ by (1).}$$

Question 1627.

(R. VAIDYANATHASWAMY) :—A surface S supposed rigid is moved by an infinitesimal rotation about an axis L into the position S′. If S′ touches S at O, prove that L must lie in one of the principal normal sections at O and pass through the centre of curvature of the other principal normal section.

Solution by P. Jagannathan.

If A is any point on the axis L and if by an infinitesimal rotation about L, any point O be carried into O′, then AO = AO′ and OO′ is perpendicular to AO.

Now let P be the point in which the axis L meets the tangent plane at O. By the infinitesimal rotation, T_0, the tangent plane at O, is carried into T_0', the tangent plane at O′ to S′. In the limit PO is the intersection of T_0 and T_0' and OO′ is perpendicular to PO so that PO and OO′ besides being conjugate directions on the surface at O are also principal directions.

Let G be the point in which the plane π containing OO′ perpendicular to PO meets L. Then GO = GO′ and OO′ is perpendicular to GO. Therefore GO and GO′ are the normals at O and O′ and G the centre of curvature in the plane π. The plane DOG is the other principal plane.

Hence the required result.

Question 1638.

(P. V. SUBBA RAO) :—Let a and b be two integers prime to each other, and N any integer. Let q_1 and r_1 be the quotient and the remainder when Na is divided by b; q_2 and r_2 the quotient and remainder when $q_1 a$ is divided by b, and so on, the quotient at each stage being multiplied by a and divided by b to get the next quotient and remainder. If this be done k times, show that

(i) If $N \leqslant b^k$, the sequence of remainders $(r_1, r_2, r_3, \ldots, r_k)$ are different for different numbers;

(ii) If $A_1, A_2, \ldots, A_k$ are values of N for which the sequence of remainders are $(1, 0, 0, 0 \ldots), (0, 1, 0, 0, \ldots), \ldots, (0, 0, \ldots, 1)$

respectively, then the number $N \leqslant bk$ for which the remainders are $(r_1, r_2, \ldots, r_k)$ is congruent to $r_1A_1 + r\,A_2 + \ldots + r_kA_k$, (mod b^k).

(iii) Show how to determine the numbers $A_1, A_2, \ldots, A_k$ in any particular case, say for $a = 4$, $b = 3$, $k = 4$,

Solution by Hansraj Gupta.

(i) If possible let N and M be two different numbers $\leqslant b^k$ having the same sequences of remainders, *viz.*, $(r_1, r_2, r_3, \ldots r_k)$.

Then we must have

$$aN = q_1b + r_1 \quad , \quad aM = q'_1b + r_1,$$

$$q_1a = q_2b + r_2 \quad , \quad q'_1a = q'_2b + r_2,$$

$$\ldots \ldots \ldots \ldots \quad , \quad \ldots \ldots \ldots \ldots$$

$$q_{k-1}a = q_kb + r_k \quad , \quad q'_{k-1}a = q'_kb + r_k.$$

$$\therefore \quad a^k(N - M) = a^{k-1}b(q_1 - q'_1), \quad = a^{k-2}b^2(q_2 - q'_2) = \ldots$$

$$= b^k(q_k - q'_k) \equiv 0. \quad (\text{mod } b^k).$$

Since a, b are relatively prime, $N - M = 0$ (mod b^k).

This contradicts the supposition, since N and M are different and not greater than bk. Hence the proposition.

(ii) We have

$$aA_m = q_{m,1}b; \quad aq_{m,1} = q_{m,2}b,$$

$$\ldots \ldots \ldots \ldots \ldots \ldots \ldots$$

$$aq_{m,m-2} = q_{m,m-1}b, \quad aq_{m,m-1} = q_{m,m}b + 1, \quad aq_{m,m} = q_{m,m+1}b,$$

$$\ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots$$

$$aq_{m,k-1} = q_{m,k}b. \qquad \ldots \text{(A)}$$

$$m = 1, 2, 3, \ldots, k,$$

$$\therefore \quad aN' = a \sum_{m=1}^{k} r_m A_m = \sum_{m=2}^{k} r_m (aA_m) + r_1 (aA_1) = b \sum_{m=1}^{k} r_m q_{m,1} + r_1$$

$$a \sum_{m=1}^{k} r_m q_{m,1} = b \sum_{m=1}^{k} r_m q_{m,2} + r_2,$$

$$\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots$$

$$a \sum_{m=1}^{k} r_m q_{m,k-1} = b \sum_{m=1}^{k} r_m q_{m,k} + r_k$$

where $$N' = \sum_{m=1}^{k} r_m A_m \qquad \ldots \quad (1)$$

Therefore the number $N \leqslant b^k$ for which the sequence of remainders is $(r_1, r_2, r_3, \ldots, r_k)$ is congruent to $\sum_{m=1}^{k} r_m A_m$ (mod b^k). ... (2)

(iii) From the set of equations (A) given above, we have

$$a^{m-1} A_m = b^{m-1} q_{m,\, m-1}; \qquad a^{k-m} q_{m,m} = b^{k-m} q_{m,\, m};$$

$\therefore$ $\quad q_{m,\, m-1} = a^{m-1} y,$ where y is an integer,

and $\quad q_{m,m} = b^{k-m} x,$ where x is an integer.

Hence $$a^m y = b^{k-m}(bx) + 1 = b^{k-m+1} x + 1 \qquad \ldots \quad (3)$$

This is an indeterminate equation of the first degree and gives y.

Now $A_m = b^{m-1} y$ gives A_m. Or A_m may be obtained directly from the equation

$$a^m A_m = b^k x + b^{m-1}. \qquad \ldots \quad (4)$$

In the particular case when $a = 4$, $b = 3$, $k = 4$, we have

$$4A_1 = 81\, x_1 + 1. \quad \therefore A_1 = 81\, S_1 + 61.$$

Similarly $A_2 = 81S_2 + 66$, from the equation $16A_2 = 81x_2 + 3$;

$A_3 = 81S_3 + 9$, ,, $64A_3 = 81x_3 + 9$;

and $A_4 = 81S_4 + 27$, ,, $256A_4 = 81x_4 + 27$.

Question 1652.

(A. A. KRISHNASWAMI AYYANGAR) :—The tangents at two points P, Q of a conic meet in T. Show that the circle passing through T and touching PQ at P or Q cuts orthogonally the director circle of the conic. Hence deduce Gaskin's theorem of which the above is a special case : " The circumcircle of a triangle self-conjugate with respect to a conic cuts the director circle orthogonally."

Solution by A. Ranganatha Rao.

It is well-known that the director circles of a system of conics touching four lines are co-axal. There are three point-pairs which are the degenerate members of the system and the director circles corresponding to these are the circles described on their joins as diameters. Now, let the four lines coalesce two by two so that we have two lines TP, TQ, where TP is one coalesced pair of lines, P being the point of ultimate intersection and TQ is the other pair of coalesced lines, Q being the point of intimate intersection. The conics now become a system of conics Σ that touch TP, TQ at P, Q respectively. Of the three point-pairs of the system, only one remains distinct, namely the point-pair P, Q, while the remaining four points constituting the other two pairs are absorbed at T. The director circles of Σ form, therefore, a co-axal system S to which belong the circle described on PQ as diameter and the point-circle T (a limiting point).

Now, let p and q be the circles which pass through T and touch PQ at P and Q respectively. It is obvious that the circles p and q are orthogonal to the circle described on PQ as diameter and since they also pass through T, they must be orthogonal to every circle of the system S. This proves the required result.

Again, the circles p and q define a co-axal system S′, every member of which is orthogonal to every member of the system S. Also the circles of S′ intersect PQ in pairs of points belonging to an involution, of which the double points are P and Q. If one pair of points in the involution be A, B, then A, B are conjugate points with respect to every conic of Σ, and hence TAB is a self-polar triangle of all those conics. Thus the circle TAB is orthogonal to the director circles of the conics of Σ, which proves Gaskin's theorem.

Also solved by S. Narayanan, M. K. Hariharan and analytically by S. P. Ranganathachar and A. L. Shaikh.

Question 1667.

(A. A. KRISHNASWAMI AYYANGAR):—Prove that

$$\int_0^x e^{-\frac{1}{2}x^2}\,dx > \sqrt{\frac{\pi}{2}}\,(1 - e^{-x^2}), \qquad (x > 0),$$

Solution by A. L. Shaikh.

The above inequality is equivalent to

$$\int_0^a e^{-x^2}\,dx > \frac{\sqrt{\pi}}{2}\,(1 - e^{-\frac{1}{2}a^2}). \qquad \ldots \quad (1)$$

For
$$\int_0^a e^{-\frac{1}{2}x^2}\,dx = \sqrt{2}\int_0^{a/\sqrt{2}} e^{-x^2}\,dx,$$

so that we have to show that

$$\sqrt{2}\int_0^{a\sqrt{2}} e^{-x^2}\,dx > \sqrt{\frac{\pi}{2}}\,(1 - e^{-a^2})$$

or replacing $a\sqrt{2}$ by a on both sides, we have to prove (1).

This is easily obtained by a method used in Gibson's *Calculus* p. 340.

Let
$$I = \int_0^a e^{-x^2}\,dx = \int_0^a e^{-y^2}\,dy$$

Then
$$I^2 = \int_0^a e^{-x^2}\,dx \times \int_0^a e^{-y^2}\,dy$$
$$= \iint e^{-(x^2+y^2)}\,dx\,dy$$

taken over the square OABC $(0 \leqslant x \leqslant a,\ 0 \leqslant y \leqslant a)$ of side a. Since the integrand is positive in this region, the above integral is greater than the same taken over the quadrant of the circle $r = \frac{a}{\sqrt{2}}$; $0 \leqslant \theta \leqslant \frac{\pi}{2}$.

$$\therefore \quad I^2 > \iint e^{-(x^2+y^2)}\,dx\,dy \quad \text{taken over the quadrant.}$$

Changing into polar co-ordinates

$$I^2 > \int_0^{a/\sqrt{2}} e^{-r^2}\,r\,dr \times \int_0^{\pi/2} d\theta = \frac{\pi}{4}\,(1 - e^{-a^2/2})$$

$$\therefore \quad I > \frac{\sqrt{\pi}}{2}\left(1 - e^{-a^2/2}\right)^{\frac{1}{2}} > \frac{\sqrt{\pi}}{2}\left(1 - e^{\ a/2^2}\right)$$

since $0 < 1 - e^{-a^2/2} < 1$.

Question 1676.

(A. A. KRISHNASWAMI AYYANGAR) :—Show that there is a unique solution to the letters A, E, G, S and the missing digits, which all make the following multiplication true. No two different letters represent the same digit.

$$
\begin{array}{cccccccc}
 & & & E & * & * & * & S \\
 & & & & & A & G & E \\
\hline
 & & * & * & * & E & G & * \\
 & & * & * & G & * & E & \\
G & E & * & * & * & A & & \\
\hline
G & A & * & * & E & * & * & *
\end{array}
$$

Solution by K. Subba Rao.

From the data given, we have

$E \geqslant 3$ ... (*a*); $G \times E < 10$. ... (*b*)

Hence $G \leqslant 2$. Also

$G \times S \equiv E \pmod{10}$. ... (*c*)

G cannot be 1, for then $S = E$. So $G = 2$.

From (*a*) and (*b*), we have $E = 3$ or 4. ... (*d*)

Now (*c*) becomes $2S \equiv E \pmod{10}$. ... (c_1)

From (*d*) and (c_1), we have $E = 4$, $S = 7$.

$A \times S \equiv A \pmod{10}$, *i.e.*, ($7A \equiv A \bmod 10$).

So $A = 5$. Now let the multiplicand be EXYZS.

From the above, we must have $X = 8$ or 9. $4Z + 2 \equiv 2 \pmod{10}$. So $Z = 0$ or 5.

If $Z = 0$, $4Y \equiv 4 \pmod{10}$, *i.e.*, $Y = 1$ or 6.

If $Z = 5$, $4Y + 2 \equiv 4 \pmod{10}$, *i.e.*, $Y = 3$ or 8.

Finally, since we have E ($=4$) in the thousands place in the product, we should have $X = 9$, $Y = 6$, $Z = 0$.

Hence the solution is unique.

Also solved by Hansraj Gupta, N. Maitra, N. K. Narasimhamurti, T. K. Raghavachari, S P. Ranganathachar and A. Ranganatha Rao.

ANNOUNCEMENTS AND NEWS.

Readers are invited to contribute to the value of this section by sending suitable news items of interest.

The following gentlemen have been elected members of the Society :—

N. P. Subramania Aiyar, M.A., Maharajah's College, Ernakulam.

K. G. Panikkar, M.A. (Hons.), Maharajah's College, Ernakulam :

J. Veeravenkiah, B.A., B. Ed., Veeresalingam High School, Rajahmundry.

The Committee of the Indian Mathematical Society has been reconstituted for the years 1934—36. The Committee consists of

President : H. G. Gharpurey, I.C.S. (Retd.), 344, Shanwarpet, Poona.

Treasurer : Prof. L. N. Subramaniam, M.A., Madras Christian College, Madras

Librarian : Prof. V. B. Naik, M.A., Fergusson College, Poona.

Other Members.

Dr. R. Vaidyanathaswamy, M.A., D.Sc., F.R.S.E.

Principal G. S. Mahajani, M.A., Ph.D.

Prof. A. C. Banerji, M.A., M.Sc., F.R.A.S.

Prof. Mukand Lal, M.A.

Principal T. K. Venkataraman, M.A.

Prof. Arunachala Sastri, M.A.

Prof. A. Narasinga Rao, M.A., L.T.

Dr. Ram Behari Lal, M.A., Ph.D.

Prof. S. B. Belekar, M A.

The Committee has passed a resolution recording its appreciation of the services rendered to the Society by the retiring President and Members of the Committee.

The Second Karnataka Astronomical Conference was held at Mysore between the 13th and 16th of October 1934 under the gracious patronage of Their Highnesses the Maharaja and the Yuvaraja of Mysore and the Presidentship of Sir Vepa Ramesam, *Kt.*, Judge, High Court of Judicature, Madras. The Yuvaraja of Mysore, in opening the Conference, urged the need for a reform of the Indian Almanac at the hands of scholars well-versed in Mathematics and Astronomy and skilled in observation and possessed of a critical mind. The President pointed out the usefulness of an Indian Observatory and appealed to the enlightened Ruler and the Dewan of Mysore to crown their constructive work by the creation of an Observatory for the study of celestial phenomena.

More than thirty papers were read in four different languages (English, Samskrit Tamil and Kannada) relating to various controversal points in practical Hindu Astronomy, such as the exact longitude of the first point of Mesha, the maximum and minimum duration of a Tithi, etc. The programme included also three public lectures on astronomical subjects, illustrated by lantern slides, an exhibition of astronomical charts, models, instruments and calculating devices, and Spectroscopic and Telescopic demonstrations arranged with the co-operation of the Physics and Mathematics Departments of the Intermediate College, Mysore.

The Conference passed a resolution urging the need for an expert Board of Studies in Hindu Astrronomy to work in consultation with the Mysore University Board of Studies in Mathematics to carry out the aims of this Conference.

Mr. A. S. Besicovitch of Cambridge and the Rev. Alfred Young, Rector of Birdbrook., Essex, have been elected Fellows of the Royal Society of London.

Dr. Adolf Fraenkel, formerly of Kiel, has been appointed Professor of Mathematics at the Hebrew University, Jerusalem.

Prof. Henri Lebesgue, of the University of Paris, has been elected a foreign Member of the Royal Society of London.

Dr. B. M. Mehrotra, M.A. (Agra), Ph D. (Liverpool), has been appointed Assistant Prof. of Mathematics at the Benares Hindu University.

Our readers will be glad to learn that the collected papers of Prof. G. A. Miller are to be published by the University of Illinois The first volume of about 500 pages is expected to be published by the middle of 1935 and will contain some original historical articles in addition to Prof. Miller's earliest contributions to Group Theory. The whole work is expected to extend to six volumes.

The following publications are now being received in exchange for those of the Indian Mathematical Society:

Annali di Mathematica pura ed Applicata, Bologna.

Sitzungsberichte der Preussiche Akedemie der Wissenschaften (Math. Phys. Klasse), Berlin.

Science Progress, A quarterly review of Scientific thought, work and affairs, published by Edward Arnold & Co., London.

H. S. Hall, author of well-known text-books in Mathematics and formerly Head of the Military and Engineering side of Clifton College, died May 3. 1934 at the age of 85.

BOOKS RECEIVED FOR REVIEW.

G. F. C. Searle: EXPERIMENTAL PHYSICS, a selection of Experiments; Cambridge University Press, 1934, pp. 363. Price 16*s*.

R. W. Gurney: ELEMENTARY QUANTUM MECHANICS. Cambridge University Press, 1934, pp. 160. Price 8*s*. 6*d*.

Sri Niwas Asthana: The Trisection of an angle leading to divide an angle into any number of equal parts, pp. 14. Price 1-8-0. Published by the Author.

QUESTIONS FOR SOLUTION.

Proposers of Questions are requested to send their own Solutions along with their Questions.

1683. (A. RANGANATHA RAO) :—D, E, F are the points of contact of the in-circle of a triangle ABC with the sides BC, CA, AB, and PP′ is any diameter of the in-circle. Prove that if any one of the following properties is true, so are the others.

(i) The pedal circle of P with respect to ABC passes through P′;

(ii) The polar axis of P with respect to ABC is perpendicular to PP′;

(iii) P′, D′, E′, F′ being the diametrically opposite points to P, D, E, F; D′P, E′P, F′P meet BC, CA, AB respectively in three collinear points;

(iv) The triad (DEF) is apolar to the triad consisting of P′ and the point P repeated twice, *i.e.*, to (P′PP).

1684. (V. RAMASWAMI AIYAR) :—From a variable point O on the θ-normal at a point P of a conic, three other θ-normals are drawn to the conic, their feet being L, M, N. Prove that the circum-circle of LMN, as O varies, forms a co-axal system whose common points are Q, the point on the conic diametrically opposite to P, and M_1 the foot of the θ-perpendicular drawn from the centre of the conic on the tangent at Q.

[*Note.*—If the θ-normal at P cut the tangent at Q in Q_1, M_1 is the mid-point of QQ_1.]

1685. (V. RAMASWAMI AIYAR) :—In the foregoing problem prove further that the sides of the triangle LMN, as O varies, envelope a parabola whose focus is M_1.

1686. (A. A. KRISHNASWAMI AYYANGAR) :—If $\int_a^b x^n f(x) = 0$ for all integral $n \geqslant 0$, is $f(x)$ identically zero?

1687. (V. RAMASWAMI AIYAR AND A. NARASINGA RAO) :— Prove that perpendicular tangents to any parabola inscribed in an equilateral triangle are also principal axes of a conic inscribed in its medial triangle.

1688. (A. A. KRISHNASWAMI AYYANGAR) :—Prove the following properties of the circles of curvature of a parabola :

(*a*) No two circles of curvature can have external contact, nor can they lie, each outside the other ;

(*b*) No circle of curvature can touch internally more than two other circles of curvature ;

(*c*) Each circle of curvature encloses two infinite sets of circles of curvature; it is also enclosed in two other infinite sets of such circles.

1689. (V. RAMASWAMI AIYAR AND A. NARASINGA RAO) :—ABCD is an ortho-centric tetrad of points and l is any straight line. Parabolas are drawn inscribed in each of the triangles ABC, BCD, CDA, DAB and touching l. Show that all the parabolas touch another line l', and that l and l' are the principal axes of conics inscribed in the medial triangle of ABCD.

1690. (V. RAMASWAMI AIYAR) :—A conic passing through the in- and ex- centres of a triangle ABC has one asymptote parallel to BC. Show that that asymptote passes through A and the other asymptote passes through A′, the point diametrically opposite to A on the circum-circle of ABC. If DEF be the medial triangle of ABC, show that the conic passes through D, and the tangent to it at D is the median AD. Show that the conic also passes through the foot of the perpendicular from A on EF ; and that the tangent to it at the point passes through the foot of the perpendicular from A′ on BC.

1691. (A. B. LAL) :—A particle is projected along the inside of a smooth surface of revolution, whose axis is vertical, with a velocity v at a point P of the surface such that the direction of motion makes an angle ψ with the meridian curve through that point. Show that the initial radius of curvature of the path of the particle is given by

$$\frac{1}{\rho^2} = \frac{1}{\rho^2} + \frac{g^2 \sin^2 \psi \sin^2 \phi}{v^4}$$

where ρ is the normal curvature at P in the direction of motion, and ϕ is the angle which the normal to the surface at P makes with the vertical.

(This is a generalisation of a problem in § 210 of Besant and Ramsay's *Dynamics*).

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